PRIMITIVE AND POISSON SPECTRA OF NON-SEMISIMPLE TWISTS OF POLYNOMIAL ALGEBRAS

by

MARY-KATHERINE BRANDL

A DISSERTATION

Presented to the Department of Mathematics
and the Graduate School of the University of Oregon
in partial fulfillment of the requirements
for the degree of
Doctor of Philosophy

August 2001

"Primitive and Poisson Spectra of Non-semisimple Twists of Polynomial Algebras," a dissertation prepared by Mary-Katherine Brandl in partial fulfillment of the requirements for the Doctor of Philosophy degree in the Department of Mathematics. This dissertation has been approved and accepted by:

Professor Brad Shelton, Chair of the Examining Committee

7/12/01

Date

Committee in charge:

Professor Brad Shelton, Chair
Professor Frank Anderson
Professor Bruce Barnes
Professor Jon Brundan
Professor Lynn Kahle

Accepted by:

Vice Provost and Dean of the Graduate School

An Abstract of the Dissertation of

Mary-Katherine Brandl for the degree of Doctor of Philosophy

in the Department of Mathematics to be taken August 2001

Title: PRIMITIVE AND POISSON SPECTRA OF NON-SEMISIMPLE TWISTS OF POLYNOMIAL ALGEBRAS

Approved: ______________________________

Professor Brad Shelton

We examine a family of twists of the complex polynomial ring on n generators by a non-semisimple automorphism. In particular, we consider the case where the automorphism is represented by a single Jordan block. The multiplication in the twist determines a Poisson structure on affine n-space. We demonstrate that the primitive ideals in the twist are parameterized by the symplectic leaves associated to this Poisson structure. Moreover, the symplectic leaves are determined by the orbits of a regular unipotent subgroup of the complex general linear group.

CURRICULUM VITAE

NAME OF AUTHOR: Mary-Katherine Brandl

PLACE OF BIRTH: Fort Belvoir, Virginia

DATE OF BIRTH: July 11, 1963

GRADUATE AND UNDERGRADUATE SCHOOLS ATTENDED:

University of Oregon
University of California, Santa Cruz

DEGREES AWARDED:

Doctor of Philosophy in Mathematics, University of Oregon, 2001
Master of Science in Mathematics, University of Oregon, 1997
Bachelor of Arts, University of California, Santa Cruz, 1995

AREAS OF SPECIAL INTEREST:

Noncommutative Ring Theory
Algebraic Geometry

PROFESSIONAL EXPERIENCE:

Graduate Teaching Assistant, Department of Mathematics,
University of Oregon, Eugene, 1995-2001

AWARDS AND HONORS:

Johnson Fellowship, Summer 1997

ACKNOWLEDGEMENTS

"For everything I learn there are two I don't understand..."

Emily Saliers

I am fortunate to be surrounded by so many exceptional people. I would like to thank my parents John and Rochelle Brandl for all of their love and support, and for believing that what I do is important. I would also like to thank Frank Anderson for his encouragement, especially during my first year at the University of Oregon, and Michaela Vancliff for her advice and instruction during the last two years. I can't imagine what graduate school would have been like without Inga Johnson and Jessica Sklar, and I know that I would not have made it without them. Finally, I want to thank Brad Shelton who has been an excellent advisor and a great friend.

TABLE OF CONTENTS

CHAPTER I

INTRODUCTION

I.1. Background of Problem

One current strategy in noncommutative ring theory is to associate geometric objects to noncommutative algebras. Algebraists have been very successful analyzing primitive ideals by considering them as geometric objects. For example, a geometric focus was used in [1] to classify algebras with nice homological properties (similar to polynomial rings) in terms of the geometric structure of a collection of graded indecomposable modules. *We refer to this geometric philosophy as noncommutative* algebraic geometry. In this dissertation, we focus on a family of twists B of the polynomial algebra $S = \mathbb{C}[x_1, \ldots, x_n]$. Our goal is to give a geometric description of the primitive spectrum of B.

We offer the following examples as motivation. The primitive ideals of the universal enveloping algebra of an algebraic solvable Lie algebra, $\mathfrak{g}$, are parametrized by the symplectic leaves in the Poisson manifold $\mathfrak{g}^*$, [2]. Furthermore, these leaves are the orbits of the adjoint algebraic group of $\mathfrak{g}$. Hodges and Levasseur use the quantum group $\mathcal{O}_q(\mathrm{SL}_n)$ to define a Poisson structure on the manifold SL_n. They then demonstrate that the primitive ideals in $\mathcal{O}_q(\mathrm{SL}_n)$ are parametrized by the symplectic leaves, [7]. In [10] M. Vancliff describes the primitive spectra of a family of twists, $B(\mathfrak{m})$, of S, parametrized by the maximal ideals of an algebra R. Each twist $B(\mathfrak{m})$ of S is determined by a semisimple automorphism $\sigma_{\mathfrak{m}}$ of $\mathbb{P}^{n-1}$. The multiplication in the twist induces a Poisson structure on $\mathbb{C}^n$. Vancliff restricts to the setting where the symplectic leaves for this Poisson structure are algebraic. She defines an associated algebraic group, G, whose orbits are the symplectic leaves. She then proves that the primitive ideals in $B(\mathfrak{m})$ are parametrized by the symplectic leaves for the Poisson structure if and only if $\sigma_{\mathfrak{m}}$ has a representative in G.

In this dissertation we extend Vancliff's results to a family of twists S^{σ} of S in which the automorphism σ is not semisimple. In particular, we consider the twist of S by an automorphism that is represented by a single Jordan block. In this setting we find that the symplectic leaves of the associated Poisson structure are always algebraic. Furthermore, we find that as in Vancliff's case, the symplectic leaves are the orbits of an algebraic group, and that the primitive ideals are parametrized by these leaves.

Much of the work in Vancliff's analysis, is due to σ having more than one eigenvalue. In her setting, the commutator of x_i and x_j is a difference of eigenvalues times $x_i x_j$, and this fact makes analyzing prime and primitive ideals straightforward. Problems only arise for certain bad combinations of eigenvalues (i.e. when ratios of differences of eigenvalues are roots of unity). We avoid these eigenvalue complications in our setting, because 1 is the only eigenvalue. On the other hand, commutators of monomials are no longer monomials, and thus it is much more difficult to analyze the prime and primitive ideals. We are required to take a different approach to the problem, and are afforded a more intricate primitive spectrum.

In Vancliff's work, the Poisson geometry is relatively straightforward to analyze because the symplectic structure is evident. In our setting, some of the symplectic leaves are evident, but we must make a careful analysis of certain differential operators to find the others.

I.2. Statement of Theorems

Let σ be the automorphism of $\mathbb{P}^{n-1}$ which is represented by the matrix with ones on the diagonal and superdiagonal, and zeros everywhere else, and let B be the twist of S by σ. In section II.1.1 we will see that B is isomorphic to a quotient of the free algebra $\mathbb{C}\langle y_1, \ldots, y_n \rangle$ by a homogeneous quadratic ideal. We identify B with this quotient, and retain the notation y_i for the image of y_i in B. The algebra B is well understood as a projective object, [11], however, we are interested in understanding B as an affine object.

This thesis is organized as follows. Chapter II gives background information pertaining to the problem. In Chapter III we investigate the primitive spectrum of the twisted algebra. Our main result is the description of the primitive ideals of B.

Theorem III.1.6. *The maximal ideals in B are the ideals $\langle y_1, \dots, y_{n-1}, y_n - \lambda \rangle$, $\lambda \in \mathbb{C}$. The remaining primitive ideals are $\langle y_1, \dots, y_{n-2} \rangle$, together with a family of homogeneous ideals. These homogeneous ideals are of the form*

$$\langle y_1, \dots, y_k, f_1, \dots f_j \rangle,$$

where $k = 0, \dots, n-3$, $j = \binom{n-k-2}{2}$, each f_i is degree 2, and each collection $\{f_1, \dots f_j\}$ is determined by a unique element of $\mathbb{C}^{n-k}$.

For notation necessary for the precise statement of Theorem III.1.6, please see Construction III.1.5. From Theorem III.1.6 we see that the non-maximal primitive ideals are parametrized by the set

$$\mathscr{P} = \{\alpha \in \mathbb{C}^{n-k} | k = 2, \dots, n\},$$

where $\mathbb{C}^0 = 1$.

In Chapter IV, we construct the Poisson structure associated to the twist. Here we define a differential operator ω, which is the key to the symplectic structure. In fact, this operator represents the crucial difference between this case and the diagonal case. Each leaf is obtained by constructing a sequence of elements $f_1, \dots, f_j$, such that $\omega f_1 = 0$, and $\omega f_i = f_{i-1}$. That is, we determine the symplectic leaves by integrating with respect to ω. After a change in variables, we recognize the two dimensional symplectic leaves as open affine subsets of classical surfaces.

Proposition IV.2.1. *The 0-dimensional symplectic leaves associated to the Poisson structure are the points* $(0, \ldots, 0, \gamma) \in \mathbb{A}^n$. *The remaining leaves are two dimensional, and each of these leaves is an open subset of the image in* $\mathbb{A}^n$ *of a Veronese surface.*

For a precise statement of Propositions IV.2.1, the reader is refered to section IV.2. After describing the Poisson structure, we note that the primitive ideals are also parametrized by $\mathscr{P}$.

Corrolary IV.2.2. *There is a natural one to one correspondence between the primitive ideals in* $B = S^{\sigma}$ *and the symplectic leaves for the symplectic structure induced by* σ.

In Chapter V we realize the two-dimensional leaves as orbits of a unipotent subgroup of the general linear group.

Proposition V.1.1. *The 2-dimensional symplectic leaves for* S^{σ} *are the orbits in* $\mathbb{A}^n$ *of a regular unipotent algebraic subgroup* G *of* $\mathrm{GL}_n(\mathbb{C})$. *Furthermore,* G *acts transitively on the 0-dimensional leaves.*

Finally, in Chapter VI, we give examples of our result, and a three dimensional twist example where the automorphism has two Jordan blocks.

CHAPTER II

PRELIMINARIES

II.1. Non-Commutative Algebra

Our primary goal is to describe the ideal structure of the twist of a polynomial algebra by a degree zero automorphism. Such an algebra is a noncommutative analogue of a homogeneous coordinate ring [1]. It is defined more simply below.

II.1.1. Twisted Algebras. Given a commutative graded k-algebra $A = \oplus A_d$, and a degree 0 automorphism σ of A, we form the **twisted algebra** A^σ, with multiplication defined on homogeneous elements by $a * b = a \cdot \sigma^r(b)$, where $r = \deg a$, and $\cdot$ denotes usual multiplication in A. This new algebra A^σ retains many of the properties of the original algebra. For example, the properties of being a domain and of being Noetherian are invariant under twisting [11]. In fact, J. Zhang has shown that twisting defines an equivalence relation on the category of graded k-algebras that is analogous to Morita equivalence, in the following sense. Let $\mathrm{Gr} - A$ be the category of graded A-modules, with morphisms being graded degree 0 homomorphisms. Then a graded k-algebra B is a twisted algebra of A if and only if the categories $\mathrm{Gr} - A$ and $\mathrm{Gr} - B$ are equivalent if and only if the categories $\mathrm{Gr} - A$ and $\mathrm{Gr} - B$ are isomorphic.

Let $B = A^\sigma$. Since $B = A$ as sets, each element of $f \in B$ is also an element of A. We will write $f^0 \in A$ when f is viewed as an element of A. For an ideal I in B, let $I^0 = \{f^0 | f \in I\}$. For homogeneous $F \in I_i^0$ and $G \in A$, $FG = F * \sigma^{-i}(G) \in I^0$. It follows that if I is a homogeneous ideal in B, then I^0 is a homogeneous ideal in A. If in addition, $(I^0)^\sigma = I^0$, then B/I is A/I^0 as a graded vector space, with multiplication inherited from B, so in fact, $B/I \cong \left(A/I^0\right)^{\bar{\sigma}}$, where $\bar{\sigma}$ is the automorphism induced by σ.

Let $f, g \in B$ be homogeneous of degrees i and j respectively. Write $F = f^0 \in A_i$, and $G = g^0 \in A_j$, and assume that $F^\sigma = F$. Define $\tau_f(g)$ by $[\tau_f(g)]^0 = G^{\sigma^i}$. Then $[\tau_f(g) * f]^0 = G^{\sigma^i} F^{\sigma^j} = FG^{\sigma^i} = (f * g)^0$, so that $\tau_f(g) * f = f * g$. From this we see that if f is homogeneous with $(f^0)^\sigma = f^0$, then f is normal in B. We will say that $f \in B$ is **σ-invariant** if $(f^0)^\sigma = f^0$.

II.1.2. Note. For homogeneous σ-invariant element $f \in B$, τ_f is an automorphism of B. Furthermore, homogeneous σ-invariant elements of the same degree are associated to the same automorphism.

We write $(F_1, \ldots, F_d)$ for the ideal generated by the elements $F_1, \ldots, F_d$ in the commutative algebra A, and write $\langle f_1, \ldots, f_d \rangle$ for the ideal generated by $f_1, \ldots, f_d$ in the noncommutative algebra B. Let $f \in B_i$ be homogeneous and σ-invariant, and let

$I = \langle f \rangle$. Since f is normal, $I = f * B$, so

$$\begin{aligned} I^0 &= \{(f * g)^0 | g \in B\} \\ &= \{f^0 (g^0)^{\sigma^i} | g \in B\} \\ &= f^0 A. \end{aligned}$$

For $G \in A$, $(f^0 G)^\sigma = f^0 G^\sigma \in I^0$, so I^0 is σ-invariant. It follows that $B/\langle f \rangle \cong [A/(f^0)]^{\bar{\sigma}}$, where where $\bar{\sigma}$ is the automorphism induced by σ.

II.1.3. Notes.

1. The preceding paragraph shows that if f is σ-invariant and irreducible, then the ideal $\langle f \rangle$ is prime.

2. Let f be σ-invariant, and let P be a prime ideal in B with $f \notin P$. Since f is normal, $\langle f \rangle = B * f$. If $g \in B$, with $f * g \in P$, then $\langle f \rangle * \langle g \rangle = B * f * B * g * B = B * f * g * B = \langle g * f \rangle \subseteq P$. But then $g \in P$. It follows that f is regular modulo P.

Now, let $S = S^n = \mathbb{C}[x_1, \ldots x_n]$ be the polynomial algebra in n variables over the complex numbers, with grading given by $\deg(x_i) = 1$. A graded automorphism σ of S^n is determined by its restriction to the vector space S_1^n of degree one elements, so σ is represented by an upper-triangular $(n \times n)$-matrix. Furthermore, scalar multiples of this matrix give rise to isomorphic twisted algebras, so we can take σ to be an automorphism of $\mathbb{P}^{n-1}$.

II.1.4. Primitive Ideals. Let R be a ring. A module M_R is **faithful** if $\mathrm{Ann}_R(M) = 0$, that is, if $r \in R$ with $Mr = 0$, then $r = 0$. We say that R is **(left) right primitive**

if R has a simple faithful (left) right module. Although a right primitive ring need not be left primitive, we will usually omit the word 'right'. An ideal P in R is **primitive** if R/P is a primitive ring. A primitive ideal is prime, and each maximal ideal is primitive [5]. Furthermore, in a commutative ring an ideal is primitive if and only if it is maximal. It is not surprising then that the primitive ideals in a non-commutative ring play a role analogous to that of maximal ideals in a commutative ring.

A ring R has the **endomorphism property** if for every simple $R[z]$ module, M, $\mathrm{End}(M)$ is algebraic over k. If k is an uncountable field, and R is a countably generated k algebra, then R has the endomorphism property [9].

Proposition II.1.5. *Let k be an uncountable algebraically closed field, and let R be a primitive k-algebra. Then the center of the quotient ring $Q(R)$ is k.*

Proof. Let Z be the center of $Q(R)$, and $z \in Z$. Write $z = rs^{-1}$, with $r, s \in R$, and s regular. Since z is central, it follows that for each $p = p(z) \in R[z]$, $ps^n \in R$, where n is the z degree of p. Let L be a simple faithful R-module, and let $\bar{L} = L \otimes_R R[z]$. We claim that $\bar{L}$ is a simple faithful $R[z]$-module. As an R-module, $\bar{L} = \sum_{i \in \mathbb{Z}} L$, is a sum of faithful modules, so $\mathrm{Ann}_{R[z]}(\bar{L}) \cap R = 0$. But R is essential in $R[z]$, so $\mathrm{Ann}_{R[z]}(\bar{L}) = 0$. Now, suppose that A is a nonzero $R[z]$-submodule of $\bar{L}$, and let u be nonzero in A. Write $u = x \otimes p$, where $p = \sum_{i=0}^{n} \alpha_i z^i$. The R-module uR is contained is the module $\sum_{i=0}^{n} Lz^i$, whose simple factors are all isomorphic to L. Since L is faithful, $\mathrm{Ann}_L(s^n) \neq L$, so there is a nonzero element $v \in uR$ such that $vs^n \neq 0$. Write

$v = y \otimes q$ where $q \in R[z]$ has z degree less than or equal to n. Then $vs^n = yqs^n \otimes 1$ is in $uR \subset A$ and generates A, so $\bar{L}$ is in fact simple. Now, $R[z]$ has the endomorphism property, so $\mathrm{End}(\bar{L})$ is algebraic over k, hence equal to k. But multiplication by z is an endomorphism on $\bar{L}$, so z acts as λ for some $\lambda \in k$. But $\bar{L}$ is faithful, so $z = \lambda \in k$, and we are done. □

II.1.6. Note. In the proof of Proposition II.1.5, we actually showed that if R and $R[z]$ are primitive algebras over an uncountable algebraically closed field, then $R[z] = R$.

II.1.7. Remark. A regular normal element r in a ring R determines an automorphism φ_r of R by $xr = r\varphi_r(x)$. Suppose R is a primitive k-algebra. If r and s are elements of R that determine the same automorphism, then the element rs^{-1} is central in the quotient ring $Q(R)$. Then by Proposition II.1.5, $r = cs$ for some $c \in k$.

II.2. Poisson Geometry

II.2.1. Poisson Manifolds. Let A be a $\mathbb{C}$-algebra. A **Poisson bracket** on A is a Lie bracket $\{,\}$ on A that is a derivation in each variable. So $\{,\}$ is a skew-symmetric bilinear form that satisfies

$$(i)\quad \{x,\{y,z\}\} + \{y,\{z,x\}\} + \{z,\{x,y\}\} = 0; \text{ and}$$

$$(ii)\quad \{x,yz\} = y\{x,z\} + \{x,y\}z.$$

The pair $(A, \{,\})$ is called a **Poisson algebra**. An ideal I in A is a **Poisson ideal** if $\{I, A\} \subseteq I$, and an element $f \in A$ is a **Poisson element** if (f) is a Poisson ideal in A. Let M be a differentiable complex manifold. A **Poisson structure** on M is determined by choosing a Poisson bracket $\{,\}$ from $C^\infty(M, \mathbb{C}) \times C^\infty(M, \mathbb{C}) \to C^\infty(M, \mathbb{C})$. The pair $(M, \{,\})$ is called a **Poisson manifold.**

For any Poisson manifold $(M, \{,\})$, there is a unique differentiable field Λ of twice contravariant, skew-symmetric tensors such that for any pair $f, g \in C^\infty(M, \mathbb{C})$,

$$\{f, g\} = \Lambda(df, dg).$$

For a point $x \in M$, the rank of the 2-tensor $\Lambda(x)$ is called the **rank** of the Poisson structure at x. A **symplectic leaf** is a maximal connected Poisson submanifold N of M such that the rank of the Poisson structure at each point of N is equal to the dimension of N. By standard theory, the symplectic leaves have even dimension, and M is a disjoint union of symplectic leaves [8]. The collection of symplectic leaves is called a **foliation** of M, and we say that M is **foliated** by its symplectic leaves. Suppose that $(M, \{,\})$ is a Poisson manifold, with $M = \mathbb{C}^n$. Then the bracket $\{,\}$ is determined by its restriction to $S = \mathbb{C}[x_1, \dots, x_n]$, [8]. If in fact the bracket maps $\mathbb{C}[x_1, \dots, x_n] \times \mathbb{C}[x_1, \dots, x_n]$ into $\mathbb{C}[x_1, \dots, x_n]$, then we may determine the Poisson structure by studying the Poisson algebra $(S, \{,\})$.

II.2.2. Drinfel'd. Let S be the polynomial algebra on n generators over $\mathbb{C}$. The Poisson bracket due to Drinfel'd is defined as follows. Let R be a commutative k-algebra which is a PID but not a field, and let A be an R-algebra. Further assume that A is flat as an R-module, and that there exists a maximal ideal $\mathfrak{m}_0 = (H)$ of R which is unique with the property that $A/\langle \mathfrak{m}_0 \rangle \cong S$. For $F, G \in S$ choose preimages $\widetilde{F}, \widetilde{G} \in A$, and define the bracket of F and G to be

$$\{F, G\} = \frac{\widetilde{F}\widetilde{G} - \widetilde{G}\widetilde{F}}{H} \bmod \langle H \rangle.$$

Then $\{,\}$ is a Poisson bracket on S [3].

CHAPTER III

THE TWISTED ALGEBRA

III.1. Non-Semisimple Twists

III.1.1. The Twisted Algebra. In [10], Vancliff describes geometrically the primitive spectrum of the twist of a polynomial algebra by a diagonalizable automorphism. We are interested in the case where the automorphism is not diagonalizable. In particular, we present the case where the automorphism is represented by the Jordan block with ones on the diagonal and on the superdiagonal. Let $S = S^n$ be the polynomial algebra with n variables over the complex numbers, and let

$$\sigma = \begin{pmatrix} 1 & 1 & & & \\ & 1 & 1 & & \\ & & \ddots & \ddots & \\ & & & 1 & 1 \\ & & & & 1 \end{pmatrix}.$$

Then using the convention that $x_0 = 0$, and writing F^σ for $\sigma(F)$, we have $x_i^\sigma = x_i + x_{i-1}$. The twisted algebra $B^n = S^\sigma$ has multiplication

$$x_i * x_j = x_i x_j + x_i x_{j-1}.$$

Notice that for each $i \leq n$, we have an embedding $B^i \hookrightarrow B^n$ given by $x_i \mapsto x_i$.

To avoid the $*$ notation, we write $y_{i_1}y_{i_2}\cdots y_{i_t}$ for the element $x_{i_1} * x_{i_2} * \cdots * x_{i_t}$. Then B^n is a quotient of the free algebra $\mathbb{C}\langle y_1, \ldots, y_n\rangle$ by a homogeneous quadratic ideal. Recall that each $f \in B^n$ corresponds to a unique polynomial $f^0 \in S$. For example, $y_i^0 = x_i$, and $(y_iy_j)^0 = x_ix_j^\sigma = x_ix_j + x_ix_{j-1}$.

III.1.2. Remark: The goal is to describe the primitive ideal structure of B^n. The element y_1 is homogeneous and σ-invariant, so by III.1.1, $B^n/\langle y_1\rangle \cong B^{n-1}$. By induction, we will understand the primitive ideal structure of B^n once we describe the primitive ideals in each B^i, $i \leq n$, that do not contain y_1.

III.1.3. Example. Let $n = 2$, so $B^2 = \mathbb{C}\langle y_1, y_2\rangle/\langle y_1y_2 - y_2y_1 - y_1^2\rangle$. We will show in Lemma III.2.3 that every primitive ideal in B^n contains a homogeneous, σ-invariant element. Suppose $F = \sum_{j=0}^{d} \alpha_j x_1^{d-j}x_2^j \in S^2$ is σ-invariant.

$$\begin{aligned}
F^\sigma - F &= \sum_{j=1}^{d} \alpha_j x_1^{d-j}[(x_2^j)^\sigma - x_2^j] \\
&= \sum_{j=1}^{d} \alpha_j x_1^{d-j} \sum_{i=0}^{j-1} \binom{j}{i} x_1^{j-i}x_2^i \\
&= \sum_{j=1}^{d}\sum_{i=0}^{j-1} \alpha_j \binom{j}{i} x_1^{d-i}x_2^i \\
&= \sum_{i=0}^{d-1}\sum_{j=i+1}^{d} \alpha_j \binom{j}{i} x_1^{d-i}x_2^i.
\end{aligned}$$

Then for each i, $\sum_{j=i+1}^{d} \alpha_j \binom{j}{i} = 0$, and it follows that $\alpha_j = 0$ for $j = 1, \ldots, d$. This means that the only homogeneous σ-invariant elements of B^2 are powers of y_1. But y_1 is normal, so every non-zero primitive ideal contains y_1. The primitive ideals in the commutative algebra $B^2/\langle y_1\rangle = \mathbb{C}[y_2]$ are the maximal ideals $\langle y_2 - \gamma\rangle$, $\gamma \in \mathbb{C}$, so the non-zero primitive ideals in B^2 are the ideals $\langle y_1, y_2 - \gamma\rangle$, $\gamma \in \mathbb{C}$. Finally, 0 is a prime ideal which is not the intersection of strictly larger primitive ideals, so 0 itself must be primitive [9]. Thus the primitive ideals in B^2 are $\langle 0\rangle$, and $\langle y_1, y_2 - \gamma\rangle$ $\gamma \in \mathbb{C}$.

III.1.4. Note. From Example III.1.3, we see that for each n, the primitive ideals in B^n that contain $y_1, \dots y_{n-2}$ are $\langle y_1, \dots, y_{n-2}\rangle$, and $\langle y_1, \dots, y_{n-1}, y_n - \gamma\rangle$, $\gamma \in \mathbb{C}$. Moreover, the ideal $\langle y_1, \dots, y_{n-1}\rangle$ is prime but not primitive.

III.1.5. Construction.

Let $n > 2$, and $\alpha = (\alpha_1, \dots, \alpha_{n-2}) \in \mathbb{C}^{n-2}$. For each $j = 1, \dots, n-2$, let

$$f_\alpha^j = \left[\sum_{i=1}^{j} \alpha_{j-i+1} y_1 y_i\right] + (j-1) y_1 y_{j+1} + (j+1) y_1 y_{j+2} - y_2 y_{j+1},$$

and let $I_\alpha = \langle f_\alpha^1, \dots, f_\alpha^{n-2}\rangle$. We want to show that every primitive ideal in B^n that does not contain y_1, contains I_α, for some $\alpha \in \mathbb{C}^{n-2}$. In fact, we will show that if P is primitive with $y_1 \notin P$ then there is a unique $\alpha \in \mathbb{C}^{n-2}$ so that $I_\alpha \subset P$. Let $g_\alpha^1 = \alpha_1 y_1^2 + 2y_1 y_3 - y_2^2$, and for $j \geq 2$, let

$$g_\alpha^j = \left[\sum_{k=1}^{j}\sum_{i=1}^{k} (-1)^{k-i} \alpha_i y_1 y_{j-k+1}\right] + \left[\sum_{i=3}^{j+1} (-1)^{j-i} y_1 y_i\right] + (j+1) y_1 y_{j+2} + \left[\sum_{i=2}^{j+1} (-1)^{j-i} y_2 y_i\right].$$

Then $g_\alpha^1 = f_\alpha^1$, and

$$\begin{aligned}
g_\alpha^j + g_\alpha^{j-1} &= \sum_{k=1}^{j}\sum_{i=1}^{k}(-1)^{k-i}\alpha_i y_1 y_{j-k+1} + \sum_{k=1}^{j-1}\sum_{i=1}^{k}(-1)^{k-i}\alpha_i y_1 y_{j-k} \\
&\quad -y_1y_{j+1} + (j+1)y_1y_{j+2} + jy_1y_{j+1} - y_2y_{j+1} \\
&= \sum_{k=1}^{j}\sum_{i=1}^{k}(-1)^{k-i}\alpha_i y_1 y_{j-k+1} + \sum_{k=2}^{j}\sum_{i=1}^{k-1}(-1)^{k-i-1}\alpha_i y_1 y_{j-k+1} \\
&\quad (j-1)y_1y_{j+1} + (j+1)y_1y_{j+2} - y_2y_{j+1} \\
&= \alpha_1 y_1 y_j + \sum_{k=2}^{j}\alpha_k y_1 y_{j-k+1} + (j-1)y_1y_{j+1} + (j+1)y_1y_{j+2} - y_2y_{j+1} \\
&= \sum_{k=1}^{j}\alpha_k y_1 y_{j-k+1} + (j-1)y_1y_{j+1} + (j+1)y_1y_{j+2} - y_2y_{j+1} \\
&= f_\alpha^j.
\end{aligned}$$

Then $\langle f_\alpha^1, \ldots, f_\alpha^{n-2}\rangle = \langle g_\alpha^1, \ldots, g_\alpha^{n-2}\rangle$. Let $G_\alpha^0 = 0$, and for $j = 1, \ldots n-2$, let $G_\alpha^j = (g_\alpha^j)^0 \in S$. Then $G_\alpha^1 = \alpha_1 x_1^2 + x_1x_2 + 2x_1x_3 - x_2^2$, and for $j \geq 2$,

$$\begin{aligned}
G_\alpha^j = \left[\sum_{k=1}^{j}\sum_{i=1}^{k}(-1)^{k-i}\alpha_i x_1 x_{j-k+1}^\sigma\right] + \left[\sum_{i=3}^{j+1}(-1)^{j-i}x_1x_i^\sigma\right] \\
+ (j+1)x_1x_{j+2}^\sigma\left[\sum_{i=2}^{j+1}(-1)^{j-i}x_2x_i^\sigma\right].
\end{aligned}$$

We have

$$
\begin{aligned}
G_\alpha^j &= \left[\sum_{k=1}^{j}\sum_{i=1}^{k}(-1)^{k-i}\alpha_i x_1(x_{j-k+1}+x_{j-k})\right] \\
&\quad + \left[\sum_{i=3}^{j+1}(-1)^{j-i}x_1(x_i+x_{i-1})\right] + (j+1)x_1(x_{j+2}+x_{j+1}) \\
&\quad + \left[\sum_{i=2}^{j+1}(-1)^{j-i}x_2(x_i+x_{i-1})\right] \\
&= \left[\sum_{k=1}^{j}\sum_{i=1}^{k}(-1)^{k-i}\alpha_i x_1 x_{j-k+1}\right] + \left[\sum_{k=1}^{j-1}\sum_{i=1}^{k}(-1)^{k-i}\alpha_i x_1 x_{j-k}\right] \\
&\quad + \left[\sum_{i=3}^{j+1}(-1)^{j-i}x_1 x_i\right] + \left[\sum_{i=3}^{j+1}(-1)^{j-i}x_1 x_{i-1})\right] + (j+1)x_1x_{j+2} \\
&\quad +(j+1)x_1x_{j+1} + \left[\sum_{i=2}^{j+1}(-1)^{j-i}x_2 x_i\right] + \left[\sum_{i=2}^{j+1}(-1)^{j-i}x_2 x_{i-1}\right] \\
&= \left[\sum_{k=0}^{j-1}\sum_{i=1}^{k+1}(-1)^{k-i+1}\alpha_i x_1 x_{j-k}\right] + \left[\sum_{k=1}^{j-1}\sum_{i=1}^{k}(-1)^{k-i}\alpha_i x_1 x_{j-k}\right] \\
&\quad + \left[\sum_{i=3}^{j+1}(-1)^{j-i}x_1 x_i\right] + \left[\sum_{i=2}^{j}(-1)^{j-i-1}x_1 x_i)\right] + (j+1)x_1x_{j+2} \\
&\quad +(j+1)x_1x_{j+1} + \left[\sum_{i=2}^{j+1}(-1)^{j-i}x_2 x_i\right] + \left[\sum_{i=1}^{j}(-1)^{j-i-1}x_2 x_i\right] \\
&= \sum_{k=1}^{j-1}\left[\sum_{i=1}^{k+1}(-1)^{k-i+1}\alpha_i x_1 x_{j-k} + \sum_{i=1}^{k}(-1)^{k-i}\alpha_i x_1 x_{j-k}\right] + \alpha_1 x_1 x_j \\
&\quad -x_1x_{j+1} + (-1)^{j-1}x_1x_2 + (j+1)x_1x_{j+2} + (j+1)x_1x_{j+1} \\
&\quad -x_2x_{j+1} + (-1)^{j-2}x_2x_1 \\
&= \left[\sum_{k=1}^{j-1}\alpha_{k+1}x_1x_{j-k}\right] + \alpha_1x_1x_j + jx_1x_{j+1} + (j+1)x_1x_{j+2} - x_2x_{j+1} \\
&= \left[\sum_{i=1}^{j}\alpha_{j-i+1}x_1x_i\right] + jx_1x_{j+1} + (j+1)x_1x_{j+2} - x_2x_{j+1}.
\end{aligned}
$$

Now,

$$
\begin{aligned}
(G_\alpha^j)^\sigma - G_\alpha^j &= \sum_{i=1}^{j} \alpha_{j-i+1} x_1 (x_i^\sigma - x_i) + j x_1 (x_{j+1}^\sigma - x_{j+1}) \\
&\quad + (j+1) x_1 (x_{j+2}^\sigma - x_{j+2}) - (x_2^\sigma x_{j+1}^\sigma - x_2 x_{j+1}) \\
&= \sum_{i=1}^{j} \alpha_{j-i+1} x_1 x_{i-1} + j x_1 x_j + (j+1) x_1 x_{j+1} \\
&\quad - (x_1 + x_2)(x_j + x_{j+1}) + x_2 x_{j+1} \\
&= \sum_{i=1}^{j} \alpha_{j-i+1} x_1 x_{i-1} + j x_1 x_j + (j+1) x_1 x_{j+1} - x_1 x_j - x_1 x_{j+1} \\
&\quad - x_2 x_j - x_2 x_{j+1} + x_2 x_{j+1} \\
&= \sum_{i=2}^{j} \alpha_{j-i+1} x_1 x_{i-1} + (j-1) x_1 x_j + j x_1 x_{j+1} - x_2 x_j \\
&= \sum_{i=1}^{j-1} \alpha_{j-i} x_1 x_i + (j-1) x_1 x_j + j x_1 x_{j+1} - x_2 x_j \\
&= G_\alpha^{j-1}.
\end{aligned}
$$

It follows that the ideal $(G_\alpha^1, \ldots, G_\alpha^{n-2})$ is σ-invariant, and we claim that $I_\alpha^0 = (G_\alpha^1, \ldots, G_\alpha^{n-2})$. Since I_α is homogeneous, I_α^0 is homogeneous, so it suffices to show that every homogeneous element in I_α^0 lies in $(G_\alpha^1, \ldots, G_\alpha^{n-2})$. Suppose $F \in I_\alpha^0$ is homogeneous of degree t. Then $F = f^0$, with $f^0 \in I_\alpha$, and $\deg(f) = t$. Since each g_α^i is normal modulo $\langle g_\alpha^1, \ldots g_\alpha^{i-1} \rangle$, we can write $f = g_\alpha^1 f_1 + \cdots + g_\alpha^{n-2} f_{n-2}$, with $f_i \in B_{t-2}$. Then $F = G_\alpha^1 (f_1^0)^{\sigma^2} + \cdots + G_\alpha^{n-2} (f_{n-2}^0)^{\sigma^2} \in (G_\alpha^1, \ldots, G_\alpha^{n-2})$, so in fact $(G_\alpha^1, \ldots, G_\alpha^{n-2}) = I_\alpha^0$. This means that $B/I_\alpha \cong [S/(G_\alpha^1, \ldots, G_\alpha^{n-2})]^\sigma$.

Let $v_1 = x_1, v_2 = x_2$, and $v_3 = \alpha_1 x_1 + x_2 + x_3$, so $G_\alpha^1 = v_1 v_3 - v_2^2$. Assume that we have defined $v_k = \sum_{i=1}^{k} a_{ki} x_i$, $a_{kk} \neq 0$ for each $k = 1, \ldots, j+2$, and that the ideal $(G_\alpha^1, \ldots, G_\alpha^j)$ is equal to the ideal $(v_1 v_i - v_2 v_{i-1}, i = 3, \ldots, j+2)$. Note that this means that there are b_{ik}, $b_{kk} \neq 0$ so that $x_i = \sum_{k=1}^{i} b_{ik} v_k$. We have

$$\begin{aligned}
G_\alpha^{j+1} &= \sum_{i=1}^{j+1} \alpha_{j-i+2} x_1 x_i + (j+1) x_1 x_{j+2} + (j+2) x_1 x_{j+3} - x_2 x_{j+2} \\
&= v_1 \left[\left(\sum_{i=1}^{j+1} \alpha_{j-i+2} x_i \right) + (j+1) x_{j+2} + (j+2) x_{j+3} \right] - v_2 x_{j+2} \\
&= v_1 \left[\left(\sum_{i=1}^{j+1} \alpha_{j-i+2} x_i \right) + (j+1) x_{j+2} + (j+2) x_{j+3} \right] - v_2 \left(\sum_{k=1}^{j+2} b_{j+2,k} v_k \right),
\end{aligned}$$

so

$$\begin{aligned}
G_\alpha^{j+1} & - \sum_{k=1}^{j+1} b_{j+2,k} (v_1 v_{k+1} - v_2 v_k) \\
&= v_1 \left[\left(\sum_{i=1}^{j+1} \alpha_{j-i+2} x_i \right) + (j+1) x_{j+2} + (j+2) x_{j+3} \right] \\
&\quad - \left(\sum_{k=1}^{j+1} b_{j+2,k} v_1 v_{k+1} \right) - b_{j+2,j+2} v_2 v_{j+2} \\
&= v_1 \left[\left(\sum_{i=1}^{j+1} \alpha_{j-i+2} x_i \right) + (j+1) x_{j+2} + (j+2) x_{j+3} - \sum_{k=1}^{j+1} b_{j+2,k} v_{k+1} \right] \\
&\quad - b_{j+2,j+2} v_2 v_{j+2}. \\
&= v_1 \left[\left(\sum_{i=1}^{j+1} \alpha_{j-i+2} x_i \right) + (j+1) x_{j+2} + (j+2) x_{j+3} - \sum_{k=1}^{j+1} b_{j+2,k} \sum_{i=1}^{k+1} a_{k+1,i} x_i \right] \\
&\quad - b_{j+2,j+2} v_2 v_{j+2}.
\end{aligned}$$

Set

$$v_{j+3} = \frac{1}{b_{j+2,j+2}} \left[\sum_{i=1}^{j+1} \alpha_i x_{j-i+1} + (j+1) x_{j+2} + (j+2) x_{j+3} - \sum_{k=1}^{j+1} \sum_{i=1}^{k+1} b_{j+2,k} a_{k+1,i} x_i \right].$$

Then $G_\alpha^{j+1} = \left[\sum_{k=1}^{j+1} b_{j+2,k} (v_1 v_{k+1} - v_2 v_k) \right] + b_{j+2,j+2} (v_1 v_{j+3} - v_2 v_{j+2})$. By induction, we have a change of variables so that

$$I_\alpha^0 = (G_\alpha^1, \ldots, G_\alpha^{n-2}) \;\; = (v_1 v_j - v_2 v_{j-1}, j = 3, \ldots n).$$

We claim that

$$(v_1v_j - v_2v_{j-1}, j = 3, \dots n) = (v_1, v_2) \cap (v_iv_{j+1} - v_{i+1}v_j, i, j = 1, \dots, n-1).$$

Set $P_1 = (v_1, v_2)$, and $P_2 = (v_iv_{j+1} - v_{i+1}v_j, i, j = 1, \dots, n-1)$. It is clear that $(v_1v_j - v_2v_{j-1}, j = 3, \dots n) \subset P_1 \cap P_2$, so it suffices to show that $\cap_{j=3}^{n-1}\mathcal{V}(v_1v_j - v_2v_{j+1}) \subset \mathcal{V}(P_1 \cap P_2) = \mathcal{V}(P_1) \cup \mathcal{V}(P_2)$. Take $p = (p_1, \dots, p_n) \in \cap_{j=3}^{n-1}\mathcal{V}(v_1v_j - v_2v_{j+1})$, and assume $p \notin \mathcal{V}(P_1)$. If $p_1 = 0$, then since $p_1p_3 - p_2^2 = 0$, we must have $p_2 = 0$, contradicting that $p \notin \mathcal{V}(P_1)$. Then $p_1 \neq 0$. Write $p_1 = t^{n-1}$, and $p_2 = t^{n-2}u$. $p_1p_3 = p_2^2$, so $t^{n-1}p_3 = t^{2n-4}u^2$, and $p_3 = t^{n-3}u^2$. Assume $p_j = t^{n-j}u^{j-1}$. Then $p_1p_{j+1} = p_2p_j$, so $t^{n-1}p_{j+1} = t^{n-2}ut^{n-j}u^{j-1} = t^{2n-j-2}u^j$. It follows that $p_{j+1} = t^{n-j-1}u^j$. By induction, $p = (t^{n-1}, t^{n-2}u, \dots, tu^{n-2}, u^{n-1}) \in \mathcal{V}(v_iv_{j+1} - v_{i+1}v_j)$ for all i and j.

Now, P_2 is the kernel of the map $\mathbb{C}[v_1, \dots, v_n] \to \mathbb{C}[t, u]$ that sends v_i to $t^{n-i}u^{i-1}$, so P_2 is prime. Then I_α^0 has primary decomposition $I_\alpha^0 = P_1 \cap P_2$, with P_i prime, $P_1^\sigma = P_1$, and we claim that P_2 is also σ-invariant. First we note that P_2^σ is an ideal in S. In fact, since P_2 is a prime ideal, P_2^σ is also prime. Then we have

$$\begin{aligned} I_\alpha^0 &= (I_\alpha^0)^\sigma \\ &= (P_1 \cap P_2)^\sigma \\ &= P_1^\sigma \cap P_2^\sigma \\ &= P_1 \cap P_2^\sigma. \end{aligned}$$

But $I_\alpha^0 = P_1 \cap P_2$, so by the uniqueness part of primary decomposition, [4], we must have $P_2^\sigma = P_2$.

For $i = 2, \ldots n-2$, and $j = i+1, \ldots, n-1$, let $H^{ij}_\alpha = v_i v_{j+1} - v_{i+1} v_j$, and define $h^{ij}_\alpha \in B$ by $(h^{ij}_\alpha)^0 = H^{ij}_\alpha$. Set

$$P_\alpha = \langle f^1_\alpha, \ldots, f^{n-2}_\alpha, h^{ij}_\alpha, i = 2, \ldots, n, j = i+2, \ldots, n \rangle,$$

so

$$P^0_\alpha = P_2 = (G^1_\alpha, \ldots, G^{n-2}_\alpha, H^{ij}_\alpha, i = 2, \ldots, n, j = i+2, \ldots, n).$$

Then

$$B/P_\alpha \cong \left[S/(P^0_\alpha)\right]^\sigma,$$

is a twist of the algebra

$$S/(P^0_\alpha) \cong \mathbb{C}[t^{n-1}, ut^{n-2}, \ldots, u^{n-2}t, u^{n-1}],$$

so P_α is prime.

For $f = f(y_1, \ldots, y_{n-k}) \in B^{n-k} \subset B^n$, define $\mathscr{S}_k f$ to be $f(y_{k+1}, \ldots, y_n)$. We want to show:

Theorem III.1.6. *If P is a primitive ideal in B^n that does not contain y_1, then $P = P_\alpha$ for some $\alpha \in \mathbb{C}^{n-2}$. By induction, the primitive ideals in B^n are*

$\langle y_1, \ldots, y_{n-1}, y_n - \lambda \rangle, \lambda \in \mathbb{C}$;

$\langle y_1, \ldots, y_{n-2} \rangle$; and

$$\langle y_1, \ldots, y_k, \mathscr{S}_k f^1_\alpha, \ldots, \mathscr{S}_k f^{n-k-2}_\alpha, \mathscr{S}_k h^{ij}_\alpha \mid i = 2, \ldots, n-k-2, j = i+2, \ldots, n-k \rangle$$
$$k = 0, 1, \ldots, n-3; \alpha \in \mathbb{C}^{n-k-2}.$$

In the next section we establish some preliminary results, and then complete the proof of Theorem III.1.6.

III.2. Formulas

Let $\varphi = \mathrm{ad}(y_1) \in \mathrm{Aut}(B)$. Let $f \in B_d$ and $F = f^0 \in S$. Then $[\varphi(f)]^0 = x_1F^\sigma - Fx_1^{\sigma^d} = x_1(F^\sigma - F)$, so $\varphi(f) = 0$ if and only if f is σ-invariant. We have the following formulas.

Lemma III.2.1.

(i) $\varphi(y_d) = y_{d-1}y_1$.

(ii) $\varphi^{d-1}(y_d) = y_1^d$.

(iii) If $f \in B_d$ *is* σ*-invariant and* $F = f^0$, *then* $[f, y_2] = dy_1f$. *In particular,* $[y_1^d, y_2] = dy_1^{d+1}$.

(iv) $\varphi^N(fg) = \sum_{j=0}^{N} \left(\begin{array}{c} N \\ j \end{array} \right) \varphi^j(f)\varphi^{N-j}(g)$.

Proof. (i) $(\varphi(y_d))^0 = (y_1y_d - y_dy_1)^0 = x_1x_d^\sigma - x_dx_1^\sigma = x_1(x_d + x_{d-1}) - x_dx_1 = x_1x_{d-1} = (y_{d-1}y_1)^0$ (ii) Induct. If $\varphi^{d-1}(y_d) = y_1^d$, then $\varphi^d(y_{d+1}) = \varphi^{d-1}(\varphi(y_{d+1})) = \varphi^{d-1}(y_dy_1) = \varphi^{d-1}(y_d)y_1 = y_1^{d+1}$. (iii) $[f, y_2]^0 = F(x_2^{\sigma^d}) - x_2F^\sigma = F(x_2 + dx_1) - x_2F^\sigma = dx_1F = (dy_1f)^0$. (iv) This is the product rule for derivations. $\square$

III.2.2. Notes.

1. Lemma III.2.1(ii) implies that $\varphi^d(y_d) = 0$, so (iv) implies that for each $f \in B$ there exists N such that $\varphi^N(f) = 0$.

2. If P is an ideal containing $a = \lambda_0 + \sum_{i=1}^{t} \lambda_i y_i$, $\lambda_t \neq 0$, then Lemma III.2.1(ii) implies that P contains $\varphi^{t-1}(a) = y_1^t$. Thus if P is a prime ideal, and P contains a linear element, then P contains y_1.

3. If I is a σ-invariant ideal in B and $g \in B$, we will say that g is **σ-invariant modulo** I if $(g^0)^\sigma - g^0 \in I^0$. We can use the argument used in the proof of Lemma III.2.1(iii) to show that if $f \in B_j$ is σ-invariant modulo I, with $F = f^0$, then $[f, y_2] \equiv jy_1 f$ modulo I.

Lemma III.2.3. *Let I and P be prime ideals in B, such that $I \subsetneq P$, I^0 is σ-invariant, and $y_1 \notin P$. Then P contains an element that is nonzero, homogeneous, irreducible, and σ-invariant modulo I.*

Proof. Let $g \in P \setminus I$. Choose N minimal with $\varphi^N(g) \in I$, and set $f = \varphi^{N-1}(g)$. Then $\varphi(f)^0 = x_1[(f^0)^\sigma - f^0] \in I^0$, with $x_1 \notin I^0$ so f is σ-invariant modulo I. Write $f = \sum_{i=0}^{d} f_i$ with f_i homogeneous of degree i. Each f_i is σ-invariant modulo I, so by III.2.2.3, $[f, y_2] - \sum_{i=0}^{d} if_i y_1 \in I$. Then $dfy_1 - [f, y_2] \in P$ with

$$\begin{aligned} dfy_1 - [f, y_2] &\equiv dfy_1 - \sum_{i=0}^{d} if_i y_1 \text{ modulo } I \\ &\equiv \sum_{i=0}^{d-1} (d-i) f_i y_1 \text{ modulo } I. \end{aligned}$$

By induction on d, we may assume that f is homogeneous. Finally, suppose $f^0 = F_1 F_2 \cdots F_t$, with F_i irreducible. Since $(f^0)^\sigma = f^0$, σ permutes $\{F_1, \ldots, F_t\}$, so there exists s so that $F_i^{\sigma^s} = F_i$ for each i. But σ is a unipotent automorphism of each of the vector spaces S_j, so we must have $F_i^{\sigma} = F_i$. □

Theorem III.2.4. *Every prime ideal in B that does not contain y_1 is of the form $\langle g_1, g_2, \ldots, g_t \rangle$, where $g_1, g_2, \ldots, g_t$ is a regular sequence with g_i homogeneous irreducible and σ-invariant modulo $\langle g_1, \ldots g_{i-1} \rangle$.*

Proof. Let P be a primitive ideal. The ring B is prime Noetherian, so by Lemma III.2.3 it suffices to show that if I is a prime ideal in P, and g is nonzero, homogeneous, irreducible, and σ-invariant element modulo I, then g is regular, and the ideal $I + \langle g \rangle$ is prime. These follow from Notes II.1.3. □

III.3. Primitive Ideals

Lemma III.3.1. *Every primitive ideal that does not contain y_1 contains I_α for some $\alpha \in \mathbb{C}^{n-2}$.*

Proof. Let P be primitive, with $y_1 \notin P$ and let $\alpha, \beta \in \mathbb{C}^{n-2}$ with $\alpha_1 \neq \beta_1$. We want to find γ so that $f_\gamma^1 \in P$, so assume $f_\alpha^1, f_\beta^1 \notin P$. The elements f_α^1 and f_β^1 are regular and normal in B, hence regular and normal modulo the prime ideal P, (Note II.1.3.2). By Remark II.1.2, f_α^1 and f_β^1 determine the same automorphism of B/P. Then by Remark II.1.7 there exists $c \in \mathbb{C}$ so that $f_\alpha^1 - cf_\beta^1 \in P$. But $f_\alpha^1 - cf_\beta^1 = \alpha_1 y_1^2 + 2y_1y_3 - y_2^2 - c\beta_1 y_1^2 - 2cy_1y_3 - cy_2^2 = (\alpha_1 - c\beta_1)y_1^2 + 2(1-c)y_1y_3 - (1-c)y_2^2$. If

$c = 1$, then since $\alpha_1 \neq \beta_1$, P contains y_1^2 which would imply that $y_1 \in P$. Thus $c \neq 1$, and P contains f_γ^1 for every $\gamma \in \mathbb{C}^{n-2}$ with $\gamma_1 = \dfrac{\alpha_1 - c\beta_1}{1-c}$. Assume we have $\gamma_1, \ldots, \gamma_j$, such that if $\zeta \in \mathbb{C}^{n-2}$ with $\zeta_i = \gamma_i$, for $i = 1, \ldots j$, then $f_\zeta^i \in P$ for each $i = 1, \ldots, j$. Let $\alpha = (\gamma_1, \ldots, \gamma_j, \alpha_{j+1}, \ldots, \alpha_{n-2})$, and $\beta = (\gamma_1, \ldots, \gamma_j, \beta_{j+1}, \ldots, \beta_{n-2})$, with $\alpha_{j+1} \neq \beta_{j+1}$, and assume $f_\alpha^{j+1}, f_\beta^{j+1} \notin P$. f_α^{j+1} and f_β^{j+1} are σ-invariant modulo P, hence regular and normal modulo P. As above, there exists $b \neq 1$ so that $f_\alpha^{j+1} - bf_\beta^{j+1} \in P$. Set $\gamma_{j+1} = \dfrac{\alpha_{j+1} - b\beta_{j+1}}{1-b}$, so $f_\gamma^{j+1} \in P$. By induction, we can thus construct γ so that $I_\gamma \subseteq P$. $\square$

We are now ready to prove Theorem III.1.6

Proof of Theorem III.1.6. Let P be primitive in B, and assume $y_1 \notin P$. By Lemma III.3.1, P contains I_α for some α, so by Lemma III.2.3, $P = \langle I_\alpha, q_1, \ldots, q_s \rangle$, with q_i homogeneous, σ-invariant and irreducible modulo $\langle I_\alpha, q_1, \ldots, q_{i-1} \rangle$. Thus P corresponds to a prime ideal P^0 in S containing

$$(G_\alpha^1, \ldots, G_\alpha^{n-2}) = (v_1, v_2) \cap (v_i v_{j+1} - v_{i+1} v_j, i, j = 1, \ldots, n-1).$$

Then P^0 contains either (v_1, v_2) or $P_\alpha^0 = (v_i v_{j+1} - v_{i+1} v_j, i, j = 1, \ldots, n-1)$ but since $y_1 \notin P$, we must have $P_\alpha^0 \subseteq P^0$. Now P_α^0 is coheight two, so if $P^0 \neq P_\alpha^0$, then P^0 is coheight one or zero. In either case, P^0 contains a linear polynomial. But then by Note III.2.2.2, P^0 contains x_1, contradicting that $y_1 \notin P$. We have then shown that the primitive ideals in B that do not contain y_1 are of the form

$$P_\alpha = \langle f_\alpha^1, \ldots, f_\alpha^{n-2}, h_\alpha^{ij}, i = 2, \ldots, n, j = i+2, \ldots, n \rangle.$$

Now suppose P is primitive, and P contains $y_1, \ldots, y_k$, but $y_{k+1} \notin P$. Then P corresponds to a primitive ideal $\bar{P}$ in $B/\langle y_1, \ldots, y_k\rangle \cong B^{n-k}$. Under this isomorphism, image of y_{k+1} is y_1, so P corresponds to a primitive ideal in B^{n-k} that does not contain y_1. By the above, the image of P in B^{n-k} is P_α for some α. Since the preimage of $f \in B^{n-k}$ is $\mathscr{S}_k f$, we have

$$P = \langle y_1, \ldots, y_k, \mathscr{S}_k f_\alpha^1, \ldots, \mathscr{S}_k f_\alpha^{n-k-2}, \mathscr{S}_k h_\alpha^{ij}, i = 2, \ldots, n-k-2, j = i+2, \ldots, n-k\rangle.$$

□

CHAPTER IV

THE POISSON MANIFOLD

Here we describe the Poisson manifold associated to the twisted algebra B^n.

IV.1. The Poisson Bracket

Let $R = \mathbb{C}[h]$. Grade the polynomial ring $R[x_1, \dots, x_n]$ by $\deg(x_i) = 1$, and $\deg(h) = 0$. Let $A = R[x_1, \dots, x_n]^{\sigma_h}$, where σ_h is given on degree one elements in coordinates $x_1, \dots, x_n$, by right multiplication by

$$\sigma_h = \begin{pmatrix} 1 & h & & & \\ & 1 & h & & \\ & & \ddots & \ddots & \\ & & & 1 & h \\ & & & & 1 \end{pmatrix}.$$

Each element $f \in A$ corresponds to a unique polynomial $f^+ \in \mathbb{C}[h, x_1, \dots, x_n]$. Evaluating f^+ at $h = 0$ gives a polynomial in S. The map from A to S that takes f to $f^+(0, x_1, \dots, x_n)$ is a ring epimorphism, whose kernel is $\langle h \rangle$, so $A/\langle h \rangle \cong S$. Similarly, the map from A to B that takes f to the unique element $\widetilde{f} \in B$ with $f^+(1, x_1, \dots, x_n) = (\widetilde{f})^0$, is an epimorphism with kernel $\langle h - 1 \rangle$, so $A/\langle h - 1 \rangle \cong B^n$. The Drinfel'd Poisson bracket (II.2.2) on S is given by

$$\{x_i, x_j\} = \frac{x_i * x_j - x_j * x_i}{h} \mod \langle h \rangle,$$

where $*$ is multiplication in A. Again, using the convention that $x_0 = 0$, we have

$$\begin{aligned}\{x_i, x_j\} &= \frac{x_i x_j + h x_i x_{j-1} - x_j x_i - h x_j x_{i-1}}{h} \mod \langle h \rangle \\ &= x_i x_{j-1} - x_j x_{i-1} \mod \langle h \rangle.\end{aligned}$$

This yields the following formulas:

$$\{x_1, x_j\} = x_1 x_{j-1},\ j > 1;$$

$$\{x_i, x_j\} = x_i x_{j-1} - x_{i-1} x_j,\ i, j > 1.$$

We define differential operators $\omega = \sum_{j=2}^{n} x_{j-1} \frac{\partial}{\partial x_j}$, and $\theta = \sum_{j=1}^{n} x_j \frac{\partial}{\partial x_j}$, and observe that

$$\{x_i, -\} = x_i \omega - x_{i-1} \theta.$$

Note that for homogeneous $f \in S_j$, $\theta f = jf$. It follows that if I is a homogeneous ideal of S, with $\omega I \subseteq I$, then I is Poisson. In particular, the ideal (x_1) in S is Poisson, and the variety of x_1, $\mathcal{V}(x_1)$, is a Poisson submanifold of $(\mathbb{A}^n, S, \{,\})$ which is isomorphic to $(\mathbb{A}^{n-1}, S^{n-1}, \{,\})$. This means that as in the analysis of the primitive ideal structure, we can concentrate on describing the symplectic leaves that are not contained in $\mathcal{V}(x_1)$.

IV.1.1. Example. Let $n = 2$, so $S = \mathbb{C}[x_1, x_2]$. The Poisson bracket is given by $\{x_1, x_2\} = x_1^2$. A 0-dimensional symplectic leaf is the variety of a maximal ideal $\mathfrak{m}$, with $\{\mathfrak{m}, S\} \subset \mathfrak{m}$. These are the ideals $(x_1, x_2 - \gamma)$, $\gamma \in \mathbb{C}$. The form determined by $\{,\}$ has rank 2 at each $p \in \mathbb{A}^2 \setminus \mathcal{V}(x_1)$. It follows that the symplectic leaves are the points $\{(0, \gamma)\}$, $\gamma \in \mathbb{C}$ and the 2-dimensional leaf $\mathbb{A}^2 \setminus \mathcal{V}(x_1)$.

IV.1.2. Construction. Let $n > 2$. For each $\alpha = (\alpha_1, \ldots, \alpha_{n-2}) \in \mathbb{C}^{n-2}$, let

$$F_\alpha^j = \left[\sum_{i=1}^{j} \alpha_{j-i+1} x_1 x_i\right] + (j+1)x_1 x_{j+2} - x_2 x_{j+1},$$

and let Q_α be the ideal $(F_\alpha^1, \ldots, F_\alpha^{n-2})$. We have

$$\begin{aligned}\omega F_\alpha^1 &= \omega(\alpha_1 x_1^2 + 2x_1x_3 - x_2^2)\\ &= 2x_1x_2 - 2x_1x_2\\ &= 0.\end{aligned}$$

For $j > 1$,

$$\begin{aligned}\omega(F_\alpha^j) &= \left[\sum_{i=1}^{j} \alpha_{j-i+1} x_1 \omega(x_i)\right] + (j+1)x_1\omega(x_{j+2}) - \omega(x_2)x_{j+1} - x_2\omega(x_{j+1})\\ &= \left[\sum_{i=2}^{j} \alpha_{j-i+1} x_1 x_{i-1}\right] + (j+1)x_1x_{j+1} - x_1x_{j+1} - x_2x_j\\ &= \left[\sum_{i=1}^{j-1} \alpha_{j-i} x_1 x_i\right] + jx_1x_{j+1} - x_2x_j\\ &= F_\alpha^{j-1}.\end{aligned}$$

Thus each Q_α is a Poisson ideal. Furthermore, for $p = (p_1, \ldots, p_n) \in \mathbb{A}^n \setminus \mathcal{V}(x_1)$ there is a unique α so that $p \in \mathcal{V}(Q_\alpha)$: indeed, set

$$\begin{aligned}\alpha_1 &= \frac{-2p_1p_3 + p_2^2}{p_1^2}, \text{ and}\\ \alpha_j &= \frac{-\sum_{i=1}^{j-1} \alpha_{j-i}p_1p_{i+1} - (j+1)p_1p_{j+2} + p_2p_{j+1}}{p_1^2}, \ j > 1.\end{aligned}$$

Since $F_\alpha^j = \alpha_j x_1^2 + \left[\sum_{i=2}^{j} \alpha_{j-i+1} x_1 x_{i+1}\right] + (j+1)x_1x_{j+2} - x_2x_{j+1}$, $\alpha = (\alpha_1, \ldots, \alpha_{n-2})$ is the unique element of $\mathbb{C}^{n-2}$ with $p \in \mathcal{V}(Q_\alpha)$. We will show that each symplectic leaf for $\{,\}$, that is not contained in $\mathcal{V}(x_1)$, is an open subvariety of $\mathcal{V}(Q_\alpha)$ for some α.

IV.2. Symplectic Leaves

Let $V_\alpha = \mathcal{V}(Q_\alpha) \setminus \mathcal{V}(x_1)$. Then $\mathbb{A}^n \setminus \mathcal{V}(x_1)$ is the disjoint union of V_α, $\alpha \in \mathbb{C}^{n-2}$. For $f = f(x_1, \dots, x_{n-k}) \in S^{n-k} \subset S^n$, let $\mathscr{S}_k f = f(x_{k+1}, x_{k+2}, \dots, x_n)$. Then for each $\alpha \in \mathbb{C}^{n-k-2}$, let $\mathscr{S}_k Q_\alpha = (x_1, \dots, x_k, \mathscr{S}_k F_\alpha^1, \dots, \mathscr{S}_k F_\alpha^{n-2})$, and let $\mathscr{S}_k V_\alpha = \mathcal{V}(\mathscr{S}_k Q_\alpha) \setminus \mathcal{V}(x_1, \dots x_{n-1})$. We want to show:

Proposition IV.2.1. *The symplectic foliation of $\mathbb{A}^n$ associated to $\{,\}$ consists of the 0-dimensional leaves $\{(0, \dots, 0, \gamma)\}$, $\gamma \in \mathbb{C}$, and the two dimensional leaves $\mathcal{V}(x_1, \dots, x_{n-2})$ and $\mathscr{S}_k V_\alpha = \mathcal{V}(x_1, \dots, x_k, \mathscr{S}_k F_\alpha^1, \dots, \mathscr{S}_k F_\alpha^{n-k-2}) \setminus \mathcal{V}(x_1, \dots, x_{n-1})$, $k = 0, 1, \dots, n-3$, $\alpha \in \mathbb{C}^{n-k-2}$.*

As an immediate corollary, we have

Corollary IV.2.2. *There is a one to one correspondence between primitive ideals in the twisted algebra $B = S^\sigma$ and the symplectic leaves for the Poisson structure induced by σ.*

To prove Proposition IV.2.1 inductively, it suffices to show that the symplectic leaves for $\{,\}$ that are not contained in $\mathcal{V}(x_1)$ are the varieties $V_\alpha = \mathcal{V}(F_\alpha^1, \dots, F_\alpha^{n-2}) \setminus \mathcal{V}(x_1, \dots, x_{n-1})$, $\alpha \in \mathbb{C}^{n-2}$. We start by showing that each V_α is an irreducible 2-dimensional variety.

Lemma IV.2.3. *For each α, there is a change of coordinates so that*

$$V_\alpha = \{(t^{n-1}, t^{n-2}u, \dots, tu^{n-2}, u^{n-1}) | t, u \in \mathbb{C}, t \neq 0\}.$$

Proof. Let $v_1 = x_1, v_2 = x_2$, and $v_3 = \alpha_1 x_1 + 2x_3$. Then $F_\alpha^1 = v_1 v_3 - v_2^2$. Assume that for for $k = 3, \ldots, d+2$, we have defined $v_j = \sum_{i=1}^{j} c_{ji} x_i$, $c_{ji} \in \mathbb{C}$, $c_{jj} \neq 0$ so that the ideal $(F_\alpha^1, \ldots, F_\alpha^d)$ is equal to the ideal $(v_1 v_j - v_2 v_{j-1}, j = 3, \ldots, d+2)$. Note that this means that for each i, there exist e_{ij} so that $x_i = \sum_{j=1}^{i} e_{ij} v_j$. We have

$$\begin{aligned}
F_\alpha^{d+1} &= \left(\sum_{i=1}^{d+1} \alpha_{d-i+2} x_1 x_i \right) + (d+2) x_1 x_{d+3} - x_2 x_{d+2} \\
&= v_1 \left[\left(\sum_{i=1}^{d+1} \alpha_{d-i+2} x_i \right) + (d+2) x_{d+3} \right] - v_2 \sum_{j=1}^{d+2} e_{d+2,j} v_j \\
&= v_1 \left[\left(\sum_{i=1}^{d+1} \alpha_{d-i+2} x_i \right) + (d+2) x_{d+3} \right] - \sum_{j=1}^{d+2} e_{d+2,j} v_2 v_j.
\end{aligned}$$

Then

$$\begin{aligned}
F_\alpha^{d+1} &- \sum_{j=1}^{d+1} e_{d+2,j} (v_1 v_{j-1} - v_2 v_j) \\
&= v_1 \left[\left(\sum_{i=1}^{d+1} \alpha_{d-i+2} x_i \right) + (d+2) x_{d+3} \right] \\
&\qquad - \left[\sum_{j=1}^{d+1} e_{d+2,j} v_1 v_{j-1} \right] - e_{d+2,d+2} v_2 v_{d+2} \\
&= v_1 \left[\left(\sum_{i=1}^{d+1} \alpha_{d-i+2} x_i \right) + (d+2) x_{d+3} \right] \\
&\qquad - \sum_{j=1}^{d+2} e_{d+2,j} v_1 \left(\sum_{i=1}^{j-1} c_{j-1,i} x_i \right) - e_{d+2,d+2} v_2 v_{d+2} \\
&= v_1 \Bigg[\left(\sum_{i=1}^{d+1} \alpha_{d-i+2} x_i \right) + (d+2) x_{d+3} \\
&\qquad\qquad - \sum_{j=1}^{d+2} \sum_{i=1}^{j-1} e_{d+2,j} c_{j-1,i} x_i \Bigg] - e_{d+2,d+2} v_2 v_{d+2}.
\end{aligned}$$

Set $v_{d+3} = \dfrac{1}{e_{d+2,d+2}} \left[\left(\sum_{i=1}^{d+1} \alpha_{d-i+2} x_i \right) + (d+2) x_{d+3} - \sum_{j=1}^{d+2} \sum_{i=1}^{j-1} e_{d+2,j} c_{j-1,i} x_i \right]$, so

$F_\alpha^{d+1} = e_{d+2,d+2}(v_1v_{d+3} - v_2v_{d+2}) + \sum_{j=1}^{d+1} e_{d+2,j}(v_1v_{j-1} - v_2v_j)$. By induction,

$$\begin{aligned} Q_\alpha &= (v_1v_i - v_2v_{i-1}, i = 3, \ldots, n) \\ &= (v_1, v_2) \cap (v_iv_j - v_{i+1}v_{j-1} | i+1 < j). \end{aligned}$$

The variety of Q_α, with respect to coordinates $(v_1, v_2, \ldots, v_n)$ is

$$\mathcal{V}(Q_\alpha) = \mathcal{V}(v_1, v_2) \cup \{(t^{n-1}, t^{n-2}u, \ldots, tu^{n-2}, u^{n-1}) | t, u \in \mathbb{C}\}.$$

so that $V_\alpha = \{(t^{n-1}, t^{n-2}u, \ldots, tu^{n-2}, u^{n-1}) | t, u \in \mathbb{C}, t \neq 0\}$. □

We have shown that $\mathbb{A}^n \setminus \mathcal{V}(x_1)$ is a disjoint union of the 2-dimensional submanifolds V_α. To show that V_α are symplectic leaves, it remains to check that the form determined by $\{,\}$ has rank 2 on each V_α.

Consider the matrix $m = (\{x_i, x_j\})$:

$$\begin{pmatrix} 0 & x_1^2 & x_1x_2 & & \cdots & x_1x_{n-1} \\ -x_1^2 & 0 & x_2^2 - x_1x_3 & & \cdots & x_2x_{n-1} - x_1x_n \\ -x_1x_2 & -x_2^2 + x_1x_3 & 0 & & \cdots & x_3x_{n-1} - x_2x_n \\ \vdots & & & \ddots & & \vdots \\ -x_1x_{n-2} & & \cdots & & 0 & x_{n-1}^2 - x_{n-2}x_n \\ -x_1x_{n-1} & & \cdots & & -x_{n-1}^2 + x_{n-2} & 0 \end{pmatrix}$$

This matrix has rank 0 if and only if $x_i = 0$, for $i = 1, \ldots, n-1$, and it follows that the 0-dimensional leaves are the points of the form $(0, \ldots, 0, \gamma)$. Furthermore,

$$m = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} (0, x_1, x_2, \ldots, x_{n-1}) - \begin{pmatrix} 0 \\ x_1 \\ \vdots \\ x_{n-1} \end{pmatrix} (x_1, \ldots, x_n).$$

At the points not of the form $(0, \ldots, 0, \gamma)$, we see that m is a sum of rank 1 matrices, so has rank at most 2. Since the form is skew symmetric it must have even rank, 2. This completes the proof of Proposition IV.2.1.

CHAPTER V

ALGEBRAIC GROUP

V.1. Orbits in $\mathbb{A}^n$

In this section we will show that the symplectic leaves are the orbits of an algebraic subgroup of $GL_n(\mathbb{C})$. Let

$$N = \begin{pmatrix} 0 & 1 & & & 0 \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ 0 & & & & 0 \end{pmatrix} \in M_n(\mathbb{C}),$$

and let G be the regular solvable subgroup of $GL_n(\mathbb{C})$ given by

$$G = \{M(u,t) = ue^{tN} | u \in \mathbb{C}^\times, t \in \mathbb{C}\}.$$

Then G acts by right multiplication on $\mathbb{A}^n$. It is easily seen that G acts transitively on the set C of 0-dimensional leaves: $C = \{(0,\ldots,0,\gamma) | \gamma \in \mathbb{C}\}$. We want to show:

Proposition V.1.1. *The orbits in $\mathbb{A}^n$ of G are the two dimensional symplectic leaves and C.*

V.1.2. Example. Let $n = 2$, so that

$$G = \left\{ \begin{pmatrix} u & ut \\ 0 & u \end{pmatrix} \middle| u \in \mathbb{C}^\times, t \in \mathbb{C} \right\}.$$

Then the orbits of G are $\mathbb{A}^2 \setminus \mathcal{V}(x_1)$, and $\mathcal{V}(x_1)$ which are the the two-dimensional leaf and the union of the zero-dimensional leaves, respectively.

Proof of Proposition V.1.1. It suffices to prove that G acts transitively on each of the symplectic leaves not contained in $\mathcal{V}(x_1)$. Recall that these leaves are $V_\alpha = \mathcal{V}(F_\alpha^1, \ldots, F_\alpha^{n-1}) \setminus \mathcal{V}(x_1)$, where $F_\alpha^j = \sum_{i=1}^{j} \alpha_{j-i+1}x_1x_i + (j+1)x_1x_{j+2} - x_2x_{j+1}$. Let $p = (p_1, p_2, \ldots, p_n) \in V_\alpha$, and $M = M(u,t) \in G$.

$$\begin{aligned} M = M(u,t) &= ue^{tN} \\ &= u\sum_{i=0}^{n-1} \frac{1}{i!}(tN)^i \\ &= u\sum_{i=0}^{n-1} \frac{t^i}{i!}(N)^i, \end{aligned}$$

and $pN^i = (0, \ldots, 0, p_1, \ldots, p_{n-i})$. Then

$$\begin{aligned} pM &= u\sum_{i=0}^{n-1} \frac{t^i}{i!}p(N)^i \\ &= u\sum_{i=0}^{n-1} \frac{t^i}{i!}(0, \ldots, 0, p_1, \ldots, p_{n-i}), \end{aligned}$$

so the i^{th} coordinate of pM is $u\sum_{k=0}^{i-1} \frac{t^k}{k!}p_{i-k}$.

Evaluating F_α^1 at pM gives

$$\begin{aligned} F_\alpha^1(pM) &= \alpha_1u_1^2p_1^2 + 2up_1(\tfrac{1}{2}ut^2p_1 + utp_2 + up_3) - (utp_1 + up_2)^2 \\ &= u^2[\alpha_1p_1^2 + 2p_1(\tfrac{1}{2}t^2p_1 + tp_2 + p_3) - (tp_1 + p_2)^2] \\ &= u^2[\alpha_1p_1^2 + 2p_1p_3 - p_2^2] \\ &= 0, \end{aligned}$$

so $pM \in \mathcal{V}(F_\alpha^1)$. Assume that $pM \in \mathcal{V}(F_\alpha^1, \ldots, F_\alpha^{j-1})$. Note that to check that $pM \in \mathcal{V}(F_\alpha^1, \ldots, F_\alpha^j)$ it suffices to check that $b = pM(1,t) \in \mathcal{V}(F_\alpha^1, \ldots, F_\alpha^j)$. Evaluating F_α^j

at b gives

$$
\begin{aligned}
F_\alpha^j(b) &= \sum_{i=1}^{j} \alpha_{j-i+1} p_1 \sum_{k=0}^{i-1} \frac{t^k}{k!} p_{i-k} + (j+1) p_1 \sum_{k=0}^{j+1} \frac{t^k}{k!} p_{j-k+2} - (tp_1 + p_2) \sum_{k=0}^{j} \frac{t^k}{k!} p_{j-k+1} \\
&= \sum_{i=1}^{j} \sum_{k=0}^{i-1} \frac{t^k}{k!} \alpha_{j-i+1} p_1 p_{i-k} + \sum_{k=0}^{j+1} \frac{t^k}{k!} (j+1) p_1 p_{j-k+2} - \sum_{k=0}^{j} \frac{t^{k+1}}{k!} p_1 p_{j-k+1} \\
&\quad - \sum_{k=0}^{j} \frac{t^k}{k!} p_2 p_{j-k+1} \\
&= \sum_{i=1}^{j} \sum_{k=0}^{i-1} \frac{t^k}{k!} \alpha_{j-i+1} p_1 p_{i-k} + \sum_{k=0}^{j+1} \frac{t^k}{k!} (j+1) p_1 p_{j-k+2} - \sum_{k=1}^{j+1} \frac{t^k}{(k-1)!} p_1 p_{j-k+2} \\
&\quad - \sum_{k=0}^{j} \frac{t^k}{k!} p_2 p_{j-k+1} \\
&= \sum_{i=1}^{j} \sum_{k=0}^{i-1} \frac{t^k}{k!} \alpha_{j-i+1} p_1 p_{i-k} + \sum_{k=0}^{j+1} \frac{t^k}{k!} (j+1) p_1 p_{j-k+2} - \sum_{k=1}^{j+1} \frac{t^k}{k!} k p_1 p_{j-k+2} \\
&\quad - \sum_{k=0}^{j} \frac{t^k}{k!} p_2 p_{j-k+1} \\
&= \sum_{i=1}^{j} \sum_{k=0}^{i-1} \frac{t^k}{k!} \alpha_{j-i+1} p_1 p_{i-k} + \sum_{k=1}^{j+1} \frac{t^k}{k!} (j+1-k) p_1 p_{j-k+2} + (j+1) p_1 p_{j+2} \\
&\quad - \sum_{k=0}^{j} \frac{t^k}{k!} p_2 p_{j-k+1} \\
&= \sum_{i=0}^{j-1} \sum_{k=0}^{i} \frac{t^k}{k!} \alpha_{j-i} p_1 p_{i-k+1} + \sum_{k=1}^{j+1} \frac{t^k}{k!} (j+1-k) p_1 p_{j-k+2} + (j+1) p_1 p_{j+2} \\
&\quad - \sum_{k=0}^{j} \frac{t^k}{k!} p_2 p_{j-k+1} \\
&= \sum_{i=1}^{j-1} \sum_{k=1}^{i} \frac{t^k}{k!} \alpha_{j-i} p_1 p_{i-k+1} + \sum_{k=1}^{j-1} \frac{t^k}{k!} (j+1-k) p_1 p_{j-k+2} + (j+1) p_1 p_{j+2} \\
&\quad - \sum_{k=1}^{j-1} \frac{t^k}{k!} p_2 p_{j-k+1} + \sum_{i=0}^{j-1} \alpha_{j-i} p_1 p_{i+1} + \frac{t^j}{j!} p_1 p_2 + (j+1) p_1 p_{j+2} \\
&\quad - \frac{t^j}{j!} p_2 p_1 - p_2 p_{j+1} \\
&= \sum_{i=1}^{j-1} \sum_{k=1}^{i} \frac{t^k}{k!} \alpha_{j-i} p_1 p_{i-k+1} + \sum_{k=1}^{j-1} \frac{t^k}{k!} (j+1-k) p_1 p_{j-k+2} + (j+1) p_1 p_{j+2},
\end{aligned}
$$

since $\left(\sum_{i=0}^{j-1} \alpha_j p_1 p_{i+1}\right) + (j+1)p_1 p_{j+2} - p_2 p_{j+1} = F_\alpha^j(p) = 0$. Now,

$$\sum_{i=1}^{j-1} \sum_{k=1}^{i} \frac{t^k}{k!} \alpha_{j-i} p_1 p_{i-k+1} = \sum_{k=1}^{j-1} \sum_{i=1}^{j-k} \frac{t^k}{k!} \alpha_{j-k} p_1 p_i,$$

so

$$\begin{aligned} F_\alpha^j(b) &= \sum_{k=1}^{j-1} \frac{t^k}{k!} \left[\left(\sum_{i=1}^{j-k} \alpha_{j-k} p_1 p_i \right) + (j-k+1) p_1 p_{j-k+2} - p_2 p_{j-k+1} \right] \\ &= \sum_{k=1}^{j-1} \frac{t^k}{k!} F_\alpha^{j-k}(p) \\ &= 0, \end{aligned}$$

so G acts on each V_α. Now, suppose $q = (q_1, q_2, \ldots, q_n) \in V_\alpha$. We want to find u and t such that $pM(u,t) = q$. Set $u = \frac{q_1}{p_1}$, and $t = \frac{q_2}{q_1} - \frac{p_2}{p_1}$, and let $c = (c_1, \ldots, c_n) = pM(u,t)$. We have $c_1 = up_1 = q_1$, and

$$\begin{aligned} c_2 &= utp_1 + up_2 \\ &= \frac{q_1}{p_1} \left(\frac{q_2}{q_1} - \frac{p_2}{p_1} \right) p_1 + \left(\frac{q_1}{p_1} \right) p_2 \\ &= q_2 - \frac{q_1 p_2}{p_1} + \frac{q_1 p_2}{p_1} \\ &= q_2. \end{aligned}$$

Thus $pM(u,t)$ is an element of V_α whose first two coordinates are q_1 and q_2. Since each element of V_α is determined by its first two coordinates, $pM(u,t)$ must be q. □

V.2. Momentum Map

Let (M, Ω) be a symplectic manifold.

For each $f \in C^\infty(M, \mathbb{C})$, there is a differentiable vector field X_f on M such that for all $g \in C^\infty(M, \mathbb{C})$

$$X_f(g) = \{f, g\}.$$

The vector field X_f is called the **Hamiltonian vector field associated with** f, or **admitting** f **as a Hamiltonian**. Let G be a Lie group with Lie algebra $\mathfrak{g}$, and let Φ be a right action of G on M

$$\Phi(x, g) = \Phi_g(x) = x.g, \ x \in M, g \in G.$$

The action of G is **symplectic** if G acts by symplectomorphisms, that is, for each $g \in G$,

$$\Phi_g^* \Omega = \Omega.$$

For $X \in G$, the **fundamental vector field associated with** X is the vector field X_M on M defined by

$$X_M(x) = \frac{d}{ds}(x.\exp(-sX))\big|_{s=0}$$

A symplectic action Φ of a Lie group G on M is Hamiltonian if and only if there exists a differentiable map $J : M \to \mathfrak{g}^*$ such that for every $X \in \mathfrak{g}$, the associated fundamental vector field X_M admits the function J_X

$$J_X(x) = \langle J(x), X \rangle, \ x \in M$$

as a Hamiltonian, [8]. Such a map $J : M \to \mathfrak{g}^*$ is called a **momentum map of the Hamiltonian action** Φ.

V.2.1. Example. Consider the Poisson manifold $(\mathbb{A}^3, \{,\})$ determined by the twist

$$B^3 = \mathbb{C}\langle y_1, y_2, y_3\rangle / J,$$

where J is the ideal

$$J = \langle y_1y_2 - y_2y_1 - y_1^2,\ y_1y_3 - y_3y_1 - y_1y_2 + y_1^2,\ y_2y_3 - y_3y_2 - y_2^2 + y_1y_3\rangle.$$

Recall that the symplectic leaves are the orbits in $\mathbb{A}^3$ of the algebraic group

$$G = \left\{ M(u,t) = \begin{pmatrix} u & ut & \frac{1}{2}ut^2 \\ 0 & u & ut \\ 0 & 0 & u \end{pmatrix} \,\middle|\, u \in \mathbb{C}^\times, t \in \mathbb{C} \right\}.$$

Let $\mathfrak{g}$ be the Lie algebra of G, and let

$$X = \begin{pmatrix} a & t & 0 \\ 0 & a & t \\ 0 & 0 & a \end{pmatrix} \in \mathfrak{g}.$$

Then

$$\exp(-sX) = \begin{pmatrix} e^{-su} & -e^{-su}st & \frac{e^{-su}(st)^2}{2} \\ 0 & e^{-su} & -e^{-su}st \\ 0 & 0 & e^{-su} \end{pmatrix}.$$

The fundamental vector field $X_{\mathbb{A}^3}$ is given by

$$X_{\mathbb{A}^3,x}(f) \ = \frac{d}{ds}(x.\exp(-sX))\big|_{s=0}.$$

In particular,

$$\begin{aligned} X_{\mathbb{A}^3}(x_1) &= \frac{d}{ds}(e^{-su}x_1)\big|_{s=0} \\ &= -ue^{-su}x_1\big|_{s=0} \\ &= -ux_1 \\ X_{\mathbb{A}^3}(x_2) &= \frac{d}{ds}(-e^{-su}stx_1 + e^{-su}x_2)\big|_{s=0} \\ &= ue^{-su}stx_1 - e^{-su}tx_1 - ue^{-su}x_2\big|_{s=0} \\ &= -tx_1 - ux_2. \end{aligned}$$

Now, suppose that J is a momentum map for the action of G. Then there is an element $g \in S$ such that $X_g = X_{\mathbb{A}^3}$. But

$$\begin{aligned} X_g(x_1) &= \{g, x_1\} \\ &= -\{x_1, g\} \\ &= -x_1\omega g \end{aligned}$$

and

$$\begin{aligned} X_g(x_2) &= \{g, x_2\} \\ &= -\{x_2, g\} \\ &= -x_2\omega g + x_1\theta g. \end{aligned}$$

This implies that $\omega g = u$, and $\theta g = t$. This only holds for $g = 0$ and $u = t = 0$, and we conclude that there is no momentum map for the action of G.

CHAPTER VI

EXAMPLES

VI.1. Standard Examples

VI.1.1. Three Dimensional Example. Let $S = \mathbb{C}[x_1, x_2, x_3]$, and

$$\sigma = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $S^\sigma \cong B = \mathbb{C}\langle y_1, y_2, y_3\rangle / J$ where

$$J = \langle y_1y_2 - y_2y_1 - y_1^2,\ y_1y_3 - y_3y_1 - y_1y_2 + y_1^2,\ y_2y_3 - y_3y_2 - y_2^2 + y_1y_3\rangle.$$

The primitive ideals of B are

$$\langle y_1, y_2, y_3 - \gamma\rangle,\ \gamma \in \mathbb{C};$$

$$\langle y_1\rangle;\ \text{and}$$

$$\langle f_\alpha = \alpha y_1^2 + 2y_1y_3 - y_2^2\rangle,\ \alpha \in \mathbb{C}.$$

The Poisson bracket on S induced by σ is given by

$$\{x_1, x_2\} = x_1^2, \qquad \{x_1, x_3\} = x_1x_2, \qquad \{x_2, x_3\} = x_2^2 - x_1x_3.$$

The symplectic leaves associated to the bracket are

the points :	$\{(0, 0, \gamma)\}, \gamma \in \mathbb{C};$
the plane :	$\mathcal{V}(x_1) \setminus \mathcal{V}(x_1, x_2)$
and the quadratic surfaces :	$V_\alpha = \mathcal{V}(F_\alpha = \alpha x_1^2 + 2x_1x_3 - x_2^2) \setminus \mathcal{V}(x_1, x_2),\ \alpha \in \mathbb{C}.$

Let G be the algebraic group consisting of matrices of the form

$$M(u,t) = ue^{tN} = \begin{pmatrix} u & ut & \frac{1}{2}ut^2 \\ 0 & u & ut \\ 0 & 0 & u \end{pmatrix}.$$

G acts by right multiplication on $\mathbb{A}^3$. and it is easy to see that G acts transitively on the 0-dimensional leaves P_γ. The remaining orbits are the 2-dimensional leaves, V_α, $\alpha \in \mathbb{C}$.

VI.1.2. Four Dimensional Example. Let $S = \mathbb{C}[x_1, x_2, x_3, x_4]$, and

$$\sigma = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then $B = S^\sigma$ is isomorphic to $\mathbb{C}\langle y_1, y_2, y_3, y_4 \rangle / J$, where J is the ideal

$$\begin{aligned} J = \langle\ & y_1y_2 - y_2y_1 - y_1^2, \\ & y_1y_3 - y_3y_1 - y_1y_2 + y_1^2, \\ & y_1y_4 - y_4y_1 - y_1y_3 + y_1y_2 - y_1^2, \\ & y_2y_3 - y_3y_2 - y_2^2 + y_1y_3, \\ & y_2y_4 - y_4y_2 - y_2y_3 + y_2^2 - y_1y_4 + y_1y_3, \\ & y_3y_4 - y_4y_3 - y_3^2 + y_2y_4 \ \rangle \end{aligned}$$

From the 2-dimensional example, we know that the primitive ideals in B that contain y_1 are

$$\langle y_1, y_2, y_3, y_4 - \gamma \rangle;$$

$$\langle y_1, y_2 \rangle; \quad \text{and}$$

$$\langle y_1, \mathscr{S}_1 f_\alpha = \alpha y_2^2 + 2y_2y_4 - y_3^2 \rangle, \alpha \in \mathbb{C}.$$

We now compute the primitive ideals that do not contain y_1. Let

$$G_\alpha^1 = \alpha_1 x_1^2 + x_1x_2 + 2x_1x_3 - x_2^2.$$

Set $v_1 = x_1, v_2 = x_2$, and $v_3 = \alpha_1 x_1 + x_2 + 2x_3$, so $G^1_\alpha = v_1 v_3 - v_2^2$. Now let $G^2_\alpha = \alpha_2 x_1^2 + \alpha_1 x_1 x_2 + 2x_1 x_3 + 3x_1 x_4 - x_2 x_3$. Then

$$\begin{aligned} 2G^2_\alpha &= x_1(2\alpha_2 x_1 + 2\alpha_1 x_2 + 4x_3 + 6x_4) - x_2(v_3 - \alpha x_1 - x_2) \\ &= x_1(2\alpha_2 x_1 + 3\alpha_1 x_2 + 4x_3 + 6x_4) - x_2 v_3 + x_2^2, \end{aligned}$$

and

$$2G^2_\alpha + G^1_\alpha = x_1[(\alpha_1 + 2\alpha_2)x_1 + (3\alpha_1 + 1)x_2 + 6x_3 + 6x_4] - x_2 v_3.$$

Set $v_4 = (\alpha_1 + 2\alpha_2)x_1 + (3\alpha_1 + 1)x_2 + 6x_3 + 6x_4$. Then $G^2_\alpha = v_1 v_3 - v_2 v_3$. Let $H^{23}_\alpha = v_2 v_4 - v_3^2$. We have

$$\begin{aligned} H^{23}_\alpha &= v_2 v_4 - v_3^2 \\ &= x_2[(\alpha_1 + 2\alpha_2)x_1 + (3\alpha_1 + 1)x_2 + 6x_3 + 6x_4] - (\alpha_1 x_1 + x_2 + 2x_3)^2 \\ &= -\alpha_1^2 x_1^2 + (2\alpha_2 - \alpha_1)x_1 x_2 - 4\alpha_1 x_1 x_3 + 3\alpha_1 x_2^2 + 2x_2 x_3 + 6x_2 x_4 - 4x_3^2. \end{aligned}$$

Then

$$\frac{1}{2}(H^{23}_\alpha + \alpha_1 G^1_\alpha) = \alpha_2 x_1 x_2 - \alpha_1 x_1 x_3 + \alpha_1 x_2^2 + x_2 x_3 + 3x_2 x_4 - 2x_3^2.$$

Let

$$\widetilde{h^{23}_\alpha} = \alpha_2 y_1^2 - \alpha_2 y_1 y_2 + \alpha_1 y_1 y_3 - \alpha_1 y_2^2 - 3y_2 y_4 + 2y_3^2.$$

Then $(\widetilde{h^{23}_\alpha})^0 = -\frac{1}{2}(H^{23}_\alpha + \alpha_1 G^1)$, and the primitive ideals in B that do not contain y_1 are

$$\langle f^1_\alpha = \alpha_1 y_1^2 + 2y_1 y_3 - y_2^2, f^2_\alpha = \alpha_2 y_1^2 + \alpha_1 y_1 y_2 + y_1 y_3 + 3y_1 y_4 - y_2 y_3, \widetilde{h^{23}_\alpha}\rangle, \alpha \in \mathbb{C}^2.$$

The bracket on $S = \mathbb{C}[x_1, x_2, x_3, x_4]$ induced by σ is given by

$$\begin{array}{lll} \{x_1, x_2\} = x_1^2 & \{x_1, x_4\} = x_1x_3 & \{x_2, x_4\} = x_2x_3 - x_1x_4 \\ \{x_1, x_3\} = x_1x_2 & \{x_2, x_3\} = x_2^2 - x_1x_3 & \{x_3, x_4\} = x_3^2 - x_2x_4 \end{array}$$

From the two dimensional example, we know that the symplectic leaves contained in $\mathcal{V}(x_1)$ are:

$$P_\gamma = \{(0,0,0,\gamma)\},\ \gamma \in \mathbb{C};$$

$$\mathcal{V}(x_1, x_2) \setminus \mathcal{V}(x_1, x_2, x_3);\ \text{and}$$

$$\mathscr{S}_1 V_\alpha = \mathcal{V}(\mathscr{S}_1 F_\alpha = \alpha x_2^2 + 2x_2x_4 - x_3^2) \setminus \mathcal{V}(x_1, x_2, x_3),\ \alpha \in \mathbb{C}.$$

Let $\alpha \in \mathbb{C}^2$, and let

$$F_\alpha^1 = \alpha_1 x_1^2 + 2x_1x_3 - x_2^2,\ \text{and}$$

$$F_\alpha^2 = \alpha_2 x_1^2 + \alpha_1 x_1 x_2 + 3x_1x_4 - x_2x_3.$$

Then $\omega F_\alpha^2 = F_\alpha^1$, and $\omega F_\alpha^1 = 0$, so the ideal (F_α^1, F_α^2) is Poisson. The two dimensional leaves not contained in $\mathcal{V}(x_1)$ are

$$\mathcal{V}(F_\alpha^1, F_\alpha^2) \setminus \mathcal{V}(x_1, x_2, x_3),\ \alpha \in \mathbb{C}^2.$$

VI.2. More Examples

Here we consider the case where the automorphism σ is not represented by a Jordan block.

VI.2.1. Example Let

$$\sigma = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & q \end{pmatrix},$$

where $q \in \mathbb{C}$, and let $A = S^\sigma$. First we consider the case where $q = 1$. One can check that every primitive ideal contains an element of the form $\alpha x_1 + \beta x_3$. It follows that the primitive ideals are

$$\langle y_3 \rangle, \ \langle y_1, y_2 - \lambda_2, y_3 - \lambda_3 \rangle, \ \langle y_1 + \beta y_3 \rangle,$$

The bracket induced by σ is

$$\{x_1, x_2\} = x_1^2, \qquad \{x_1, x_3\} = 0, \qquad \{x_2, x_3\} = -x_1 x_3.$$

The form determined by $\{,\}$ has rank zero if and only if $x_1 = 0$, and rank 2 otherwise. It follows that the zero dimensional leaves are the points $(0, \lambda_2, \lambda_3)$. It is easily seen that x_3 is a Poisson element, so that $\mathcal{V}(x_3) \setminus \mathcal{V}(x_1)$ is a 2-dimensional symplectic leaf. Let $\beta \in \mathbb{C}$, and set $p = x_1 + \beta x_3$. Then $\{x_1, p\} = 0 = \{x_3, p\}$, and $\{x_2, p\} = -x_1^2 - \beta x_1 x_3 = -x_1 p$, so p is Poisson. It follows that the two dimensional symplectic leaves are

$$\begin{aligned}&\mathcal{V}(x_3) \setminus \mathcal{V}(x_1), \ \text{and} \\ &\mathcal{V}(\alpha x_1 + \beta x_2) \setminus \mathcal{V}(x_1), \quad \alpha, \beta \in \mathbb{C}.\end{aligned}$$

We see that the symplectic leaves are algebraic, and are in one to one correspondence with the primitive ideals in the twisted algebra.

Now suppose that q is not a root of unity. Then $A = \mathbb{C}\langle y_1, y_2, y_3 \rangle / J$ where

$$J = \langle y_1 y_2 - y_2 y_1 - y_1^2, \ y_1 y_3 - q y_3 y_1, \ y_2 y_3 - q y_3 y_2 - y_1 y_2 \rangle.$$

We want to show that the primitive ideals of A are

$$0, \ \langle y_1 \rangle, \ \langle y_3 \rangle, \ \langle y_1, y_2, y_3 - \lambda \rangle, \ \text{and} \ \langle y_1, y_2 - \lambda, y_3 \rangle.$$

It suffices to show that every primitive ideal contains either y_1 or y_3. Note that the subalgebra spanned by y_1 and y_2 is isomorphic to B^2. We will retain the notation B^2 for this subalgebra. Let $\varphi = \mathrm{ad}(y_1)$. Then $\varphi(y_2) = y_1^2$ and $\varphi^j(y_3^i) = (q - \frac{1}{q^i})^j y_1 y_3$. From Note III.2.2.1, we know that for each element f in B^2 there exists N so that $\varphi^N(f) = 0$. Let P be primitive. If P contains neither y_1 nor y_3, then P contains an element $f = \sum_{i=0}^{d} f_i y_3^i$, with $f_i \in B^2 \subset B$, and $f_0 \neq 0$, $f_d \neq 0$, i.e. $f \notin B^2$ and $f \neq \alpha y_3^d$. Then P contains $g = \varphi(f) - (1 - \frac{1}{q^d}) y_1 f$. We have

$$g = \left(\sum_{i=0}^{d-1} [\tfrac{1}{q^i}\varphi(f_i) + (\tfrac{1}{q^d} - \tfrac{1}{q^i}) y_1 f_i] y_3^i \right) + \tfrac{1}{q^d}\varphi(f_d) y_3^d.$$

Now, if $\varphi(f_i) = \lambda y_1 f_i$, with $\lambda \neq 0$, then $\varphi^N(f_i) = \lambda^N y_1 f_i \neq 0$ for all N. This is a contradiction since φ is locally nilpotent. Then since $1 - \frac{1}{q^{d-i}} \neq 0$, it follows that $\frac{1}{q^i}\varphi(f_i) + (\frac{1}{q^d} - \frac{1}{q^i}) y_1 f_i \neq 0$ for $f_i \neq 0$. By induction P contains an element

$$h = \sum_{i=0}^{d-1} h_i y_3^i + \varphi^N(f_d) y_3^d,$$

with $\varphi^N(f_d) = 0$, and $\sum_{i=0}^{d-1} h_i y_3^i \neq 0$. By induction on d, P contains an element in B^2, so by Example III.1.3, P contains y_1. We now have shown that every primitive ideal contains either y_1 or y_3. This means that the nonzero primitive ideals in A are

$$\langle y_1 \rangle, \ \langle y_3 \rangle, \ \langle y_1, y_2, y_3 - \lambda \rangle, \ \langle y_1, y_2 - \lambda, y_3 \rangle.$$

The ideal 0 is prime, but not an intersection of strictly larger primitives, so must itself be primitive.

Next we construct the Poisson structure associated to A. Set

$$\bar{\sigma} = \begin{pmatrix} 1 & h & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1+h(q-1) \end{pmatrix}.$$

We see that $S^{\bar{\sigma}}/\langle h \rangle \cong S$, and $S^{\bar{\sigma}}/\langle h-1 \rangle \cong A$. Let $\gamma = 1-q$. The Drinfel'd bracket is given by

$$\{x_1, x_2\} = \gamma x_1^2, \qquad \{x_1, x_3\} = \gamma x_1 x_3 \qquad \{x_2, x_3\} = \gamma x_2 x_3 - x_1 x_3.$$

It is easily seen that the form induced by $\{,\}$ has rank zero if and only if $x_1 = 0$ and either $x_2 = 0$ or $x_3 = 0$. The form has rank two otherwise. It follows that the zero dimensional leaves are the points $(0,0,\lambda)$ and $(0,\lambda,0)$, $\lambda \in \mathbb{C}$. Let C be the set of zero dimensional leaves. It is easily seen that the ideals (x_1) and (x_3) are Poisson, and it follows that $\mathcal{V}(x_1) \setminus C$ and $\mathcal{V}(x_2) \setminus C$ are symplectic leaves. In fact, we will show that x_1 and x_3 are the only irreducible Poisson elements. This means that the algebraic symplectic leaves are

$$(0,0,\lambda),\ (0,\lambda,0),\ \lambda \in \mathbb{C};\ \{(0,\lambda_2,\lambda_3) | \lambda_2, \lambda_3 \neq 0\};\ \text{and}\ \{(\lambda_1, 0, \lambda_3) | \lambda_1 \neq 0\}.$$

Set $\omega = x_1 \dfrac{\partial}{\partial x_2}$, $\theta_1 = x_1 \dfrac{\partial}{\partial x_1} + x_2 \dfrac{\partial}{\partial x_2}$, $\theta_2 = x_3 \dfrac{\partial}{\partial x_3}$, and $\theta = \theta_1 + \theta_2$. Then

$$\{x_1, -\} = x_1 \omega + \alpha x_1 \theta_2,$$

$$\{x_1, -\} = \alpha x_2 \theta_2 - x_1 \theta + x_2 \omega,$$

$$\{x_2, -\} = x_3 \omega - \alpha x_3 \theta_1.$$

Suppose $p \in S$ is a Poisson element with $p \neq x_1, x_3$. Since $x_1 \omega p + \alpha x_1 \theta_2 p \in (p)$, $\omega p + \alpha \theta_2 p \in (p)$. Also $x_3 \omega p - \alpha x_3 \theta_1 p \in (p)$, so $\omega p - \alpha \theta_1 p \in (p)$. Then

$$\alpha(\theta_1 + \theta_2) p = \alpha \theta p \in (p),$$

so p is homogeneous. Write $p = \sum p_i x_3^i$, with $p_i \in k[x_1, x_2]$, and $\deg(p_i) = d - i$. $\omega p + \alpha\theta_2 p \in (p)$, so there exists $\lambda \in \mathbb{C}$ so that

$$\sum(\omega p_i)x_3^i + \sum \alpha i p_i x_3^i = \sum \lambda p_i x_3^i.$$

Then for each i there exists μ_i with $\omega p_i = \mu_i p_i$. We claim that $\mu_i = 0$ for all i. If so, then $\lambda p_i = \alpha i p_i$ for all i, so $p = \alpha x_1^{d-i} x_3^i$, and we are done. To prove the claim, suppose $g \in A$ with $\omega g = \lambda g$. Write $g = \sum_{i=0}^{t} x_2^i p_i$, with $p_i \in k[x_1, x_3] \subset S$. Then

$$\omega p = \sum_{i=1}^{t} i x_1 x_2^{i-1} p_i = \sum_{i=0}^{t-1} (i+1) x_1 x_2^i p_{i+1}.$$

Then $\lambda x_2^d p_d = 0$, and $\lambda p_i = (i+1) x_1 p_{i+1}$ for each i. If $\lambda \neq 0$, then $p = 0$, so we are done.

BIBLIOGRAPHY

[1] ARTIN, M., TATE, J., AND VAN DEN BERGH, M., "Some Algebras Associated to Automorphisms of Elliptic Curves," in *The Grothendieck Festschrift*, **1** Birkhäuser, Basel, (1990), 33-85.

[2] DIXMIER, *Enveloping Algebras*, North-Holland, Amsterdam, (1977).

[3] DRINFEL'D, V.G., *Quantum Groups*, Proceedings of the International Congress of Mathematics, Berkeley, California, (1986).

[4] EISENBUD, D., *Commutative Algebra with a View Toward Algebraic Geometry*, Springer-Verlag, New York, New York, (1995).

[5] GOODEARL, K.R., AND WARFIELD, R.B., *An Introduction to Noncommutative Noetherian Rings*, London Mathematical Society, Cambridge University Press, (1989)

[6] HARTSHORNE, R., *Algebraic Geometry*, Springer-Verlag, New York, New York, (1977).

[7] HODGES, T.J., AND LEVASSEUR, T., "Primitive Ideals of $\mathbb{C}_q[SL(n)]$ ", J. Alg. **168** No. 2, (1994), 455-468.

[8] LIBERMANN, P, AND MARLE, C-M, *Symplectic Geometry and Analytical Mechanics*, D. Reidel Publishing Company, Dordrecht, Holland (1987)

[9] MCCONNELL, J.C., AND ROBSON, J.C., *Noncommutative Noetherian Rings*, Wiley-Interscience, Chichester, (1987).

[10] VANCLIFF, M., "Primitive and Poisson Spectra of Twists of Polynomial Rings", Algebras and Representation Theory. **2:**, (1999), 269-285.

[11] ZHANG, J., "Twisted Graded Algebras and Equivalences of Graded Categories", Proc. London Mathematical Society (3) 72 (1996) 281-311.